3.3 Cramer's Rule, Volume, and Linear Transformations

Cramer's Rule

For any $n \times n$ matrix A and any $\mathbf{b}$ in $\mathbb{R}^n$, let $A_i(\mathbf{b})$ be the matrix obtained from A by replacing column i by the vector $\mathbf{b}$

$$A_i(\mathbf{b}) = [\mathbf{a}_1 \quad \cdots \quad \mathbf{b} \quad \cdots \quad \mathbf{a}_n]$$

Theorem 7 Cramer's Rule
Let A be an invertible $n \times n$ matrix. For any $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \qquad i = 1, 2, \ldots, n \tag{1}$$

Example 1. Use Cramer's rule to compute the solution of the system.

$4x_1 + x_2 = 6$
$3x_1 + 2x_2 = 5$

The system is equivalent to $A\vec{x} = \vec{b}$, where $A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$.

Compute $A_1(b) = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix}$, $A_2(b) = \begin{bmatrix} 4 & 6 \\ 3 & 5 \end{bmatrix}$

$\det A = 5$, $\det A_1(b) = 7$, $\det A_2(b) = 2$

Then $x_1 = \dfrac{\det A_1(b)}{\det A} = \dfrac{7}{5}$, $x_2 = \dfrac{\det A_2(b)}{\det A} = \dfrac{2}{5}$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{7}{5} \\ \frac{2}{5} \end{bmatrix}$$

A Formula for A^{-1}

For an invertible $n \times n$ matrix A, the j-th column of A^{-1} is a vector x that satisfies

$$A\mathbf{x} = \mathbf{e}_j$$

the i-th entry of $\mathbf{x}$ is the (i, j)-entry of A^{-1}. By Cramer's rule,

$$\{(i, j)\text{-entry of } A^{-1}\} = x_i = \frac{\det A_i(\mathbf{e}_j)}{\det A} \tag{2}$$

Recall A_{ji} denotes the submatrix of A formed by deleting row j and column i, thus

$$\det A_i(\mathbf{e}_j) = (-1)^{i+j} \det A_{ji} = C_{ji}$$

Recall the cofactor $C_{ij} = (-1)^{i+j} \det A_{ij}$

where C_{ji} is a cofactor of A.
Thus

Remark: The (i,j)-entry of A^{-1} is $\frac{1}{\det A} C_{ji}$

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} \tag{3}$$

The matrix of cofactors on the right side of (3) is called the adjugate (or classical adjoint) of A, denoted by $\operatorname{adj} A$.

Theorem 8 An Inverse Formula
Let A be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

Example 2. Compute the adjugate of the given matrix, and then use Theorem 8 to give the inverse of the matrix.

$$A = \begin{bmatrix} 3 & 5 & 4 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

ANS: $\det A = 6$. We compute the cofactors:

$$C_{11} = (-1)^{1+1} \det A_{11} = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1$$

$$C_{12} = (-1)^{1+2} \det A_{12} = -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1$$

$$C_{13} = (-1)^{1+3} \det A_{13} = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1$$

$$C_{21} = -\begin{vmatrix} 5 & 4 \\ 1 & 1 \end{vmatrix} = -1$$

$$C_{22} = \begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix} = -5$$

$$C_{23} = -\begin{vmatrix} 3 & 5 \\ 2 & 1 \end{vmatrix} = 7$$

$$C_{31} = \begin{vmatrix} 5 & 4 \\ 0 & 1 \end{vmatrix} = 5$$

$$C_{32} = -\begin{vmatrix} 3 & 4 \\ 1 & 1 \end{vmatrix} = 1$$

$$C_{33} = \begin{vmatrix} 3 & 5 \\ 1 & 0 \end{vmatrix} = -5$$

$$\operatorname{adj} A = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} -1 & -1 & 5 \\ 1 & -5 & 1 \\ 1 & 7 & -5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A = \begin{bmatrix} -\frac{1}{6} & -\frac{1}{6} & \frac{5}{6} \\ \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{7}{6} & -\frac{5}{6} \end{bmatrix}$$

Remark: This method is useful if the question asks what is (i,j)-entry of A^{-1}?

$$(A^{-1})_{ij} = \frac{1}{\det A} C_{ji} \quad Eg: (A^{-1})_{23} = \frac{1}{\det A} \cdot C_{32} = \frac{1}{6}$$

Determinants as Area or Volume

Theorem 9 If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|\det A|$. If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is $|\det A|$.

Remark.

1. Let $\mathbf{a}_1$ and $\mathbf{a}_2$ be nonzero vectors. Then for any scalar c, the area of the parallelogram determined by $\mathbf{a}_1$ and $\mathbf{a}_2$ equals the area of the parallelogram determined by $\mathbf{a}_1$ and $\mathbf{a}_2 + c\mathbf{a}_1$.

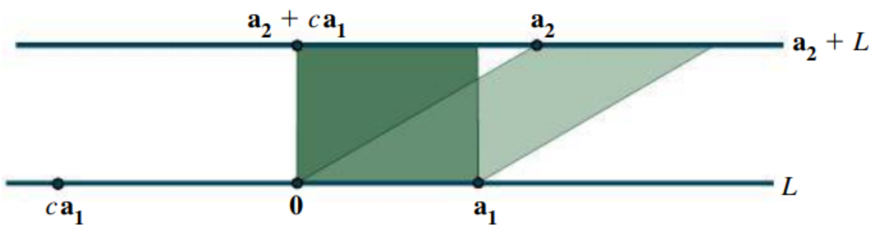


FIGURE 2 Two parallelograms of equal area.

2. An example of the parallelepiped determined by the vectors $\begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix}$, which is a cuboid.

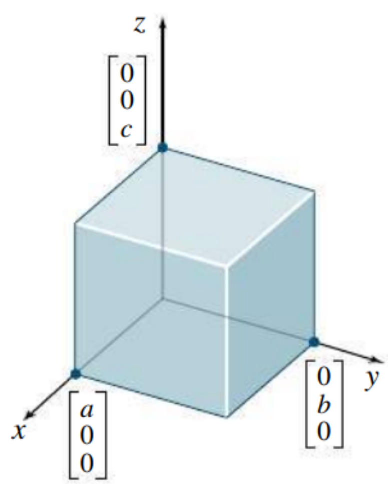


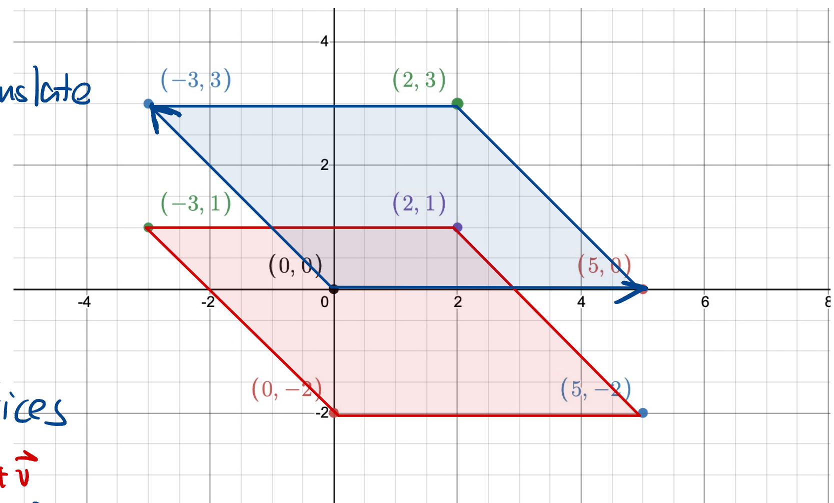
FIGURE 3
Volume $= |abc|$.

3. If one of the 3 column vectors is $\mathbf{0}$, we will have a flat parallelepiped. Note a flat parallelepiped has volume 0.

Remark: We can also compute the area by base × height $= 5 \times 3 = 15$.

Example 3. Find the area of the parallelogram whose vertices are listed.

$(0,-2),(5,-2),(-3,1),(2,1)$



ANS: To use Thm 9, we can translate one vertex to the origin. For example, substract $(0,-2)$ from each vertex to get a new parallelogram with vertices

$(0,0)$, $\underset{\vec{u}}{(5,0)}$, $\underset{\vec{v}}{(-3,3)}$, $\underset{\vec{u}+\vec{v}}{(2,3)}$

Then the new parallelogram has the same area as the original one. By Thm 9. the area is

$$\left|\det\begin{bmatrix}5 & -3\\ 0 & 3\end{bmatrix}\right| = 15$$

Linear Transformations

Theorem 10. Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then

$$\{\text{area of } T(S)\} = |\det A| \cdot \{\text{area of } S\}$$

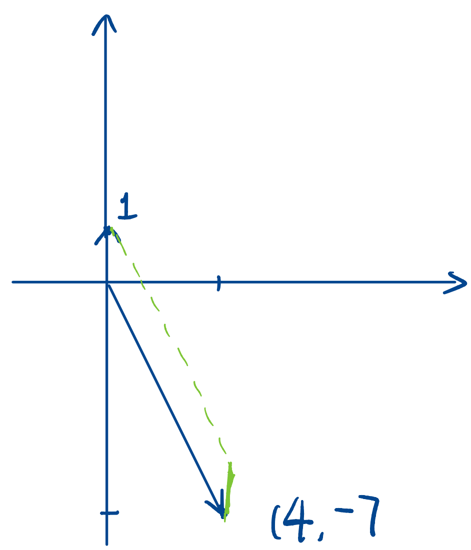
If T is determined by a 3×3 matrix A, and if S is a parallelepiped in $\mathbb{R}^3$, then

$$\{\text{volume of } T(S)\} = |\det A| \cdot \{\text{ volume of } S\}$$

Example 4. Let S be the parallelogram determined by the vectors $\mathbf{b}_1 = \begin{bmatrix}4\\-7\end{bmatrix}$, and $\mathbf{b}_2 = \begin{bmatrix}0\\1\end{bmatrix}$ and let $A = \begin{bmatrix}5 & 2\\ 1 & 1\end{bmatrix}$. Compute the area of the image of S under the mapping $\mathbf{x} \mapsto A\mathbf{x}$.

ANS: Method 1. We use Thm 10.

- $|\det A| = \left|\begin{vmatrix}5 & 2\\ 1 & 1\end{vmatrix}\right| = 3$
- $\{\text{area of } S\} \overset{\text{Thm 9}}{=\!=} \left|[\vec{b}_1 \ \ \vec{b}_2]\right| = \begin{vmatrix}4 & 0\\ -7 & 1\end{vmatrix} = 4$



Thus $\{\text{area of } A(S)\} = |\det A| \cdot \{\text{areas of } S\}$

$= 3\times 4 = 12$

Method 2. We can compute $A\vec{b}_1$ and $A\vec{b}_2$, then use Thm 9.

$$A[\vec{b}_1 \ \ \vec{b}_2] = \begin{bmatrix} 5 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ -7 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ -3 & 1 \end{bmatrix}$$

Then the area is

$$\begin{vmatrix} 6 & 2 \\ -3 & 1 \end{vmatrix} = 6+6 = 12.$$

Exercise 5. Determine the value of the parameter s for which the system has a unique solution, and describe the solution.

$3sx_1 + 5x_2 = 3$
$12x_1 + 5sx_2 = 2$

ANS. The system is equivalent to $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 3s & 5 \\ 12 & 5s \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. We compute $A_1(\mathbf{b}) = \begin{bmatrix} 3 & 5 \\ 2 & 5s \end{bmatrix}$, $A_2(\mathbf{b}) = \begin{bmatrix} 3s & 3 \\ 12 & 2 \end{bmatrix}$, $\det A_1(\mathbf{b}) = 15s - 10$, $\det A_2(\mathbf{b}) = 6s - 36$.
Since $\det A = 15s^2 - 60 = 15\left(s^2-4\right) = 0$ for $s = \pm 2$, the system will have a unique solution for all values of $s \neq \pm 2$. For such a system, the solution will be

$$x_1 = \frac{\det A_1(\mathbf{b})}{\det A} = \frac{15s-10}{15\left(s^2-4\right)} = \frac{3s-2}{3\left(s^2-4\right)},$$

$$x_2 = \frac{\det A_2(\mathbf{b})}{\det A} = \frac{6s-36}{15\left(s^2-4\right)} = \frac{2s-12}{5\left(s^2-4\right)}$$

Exercise 6. Find a formula for the area of the triangle whose vertices are $\mathbf{0}$, $\mathbf{v}_1$, and $\mathbf{v}_2$ in $\mathbb{R}^2$.

ANS. The area of the triangle will be one half of the area of the parallelogram determined by $\mathbf{v}_1$ and $\mathbf{v}$ By Theorem 9, the area of the triangle will be $(1/2)|\det A|$, where $A = [\mathbf{v}_1 \quad \mathbf{v}_2]$.

Exercise 7. Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at $(1,3,0)$, $(-2,0,2)$, and $(-1,3,-1)$.

ANS. The parallelepiped is determined by the columns of $A = \begin{bmatrix} 1 & -2 & -1 \\ 3 & 0 & 3 \\ 0 & 2 & -1 \end{bmatrix}$, so the volume of the parallelepiped is $|\det A| = |-18| = 18$.